University of Warwick

Mathematics Institute

MA262 Second Year Essay

Quaternions as Three-Dimensional Rotations

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1 Introduction and Motivation

Rotations have always been significant to human advancement and achievement. In the past, they helped astronomers understand our Universe and its structure. Mechanisms like flour mills or windmills have used the motion of a rotation to make food or generate power. Today, they are being used in the rapidly developing field of computer graphics and animation for our entertainment.

Rotations with respect to a fixed frame of reference are commonly defined using Euler rotations [13, §7.2, p105]. With this system, any three-dimensional rotation can be defined by a sequence of 3 rotations: either each about a different Cartesian axis, or by rotating about one axis, then a different one, then about the first axis again. [10, §5.4.1, p84] For example, a rotation could be defined as the sequence of transformations $R_{\gamma,z}R_{\beta,y}R_{\alpha,x}$: a rotation by α degrees about the x-axis, then β degrees about the y-axis, then γ degrees about the z-axis. [13, §7.5, p109]

This behaviour of a three-dimensional rotation can be observed during NASA's Apollo program; the Apollo spacecraft contained a platform, suspended between two gimbals.

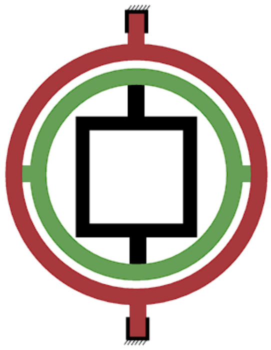

Figure 1: Diagram of the stable platform suspended in two gimbals. [8, §1, p32]

Its purpose was to keep a stable orientation, even as the spacecraft travelled to the Moon, by using sequential rotations about each axis. However, certain rotations could cause the gimbals to become coplanar; rotating about an axis perpendicular to this plane would cause the gimbals to lock. As a result, the spacecraft would lose a degree of freedom of rotation, a dangerous phenomenon NASA wanted to avoid at all costs. This phenomenon is commonly called gimbal lock. [8, §1, p31-32]

As a method of representing three-dimensional rotations, Euler rotations parametrise SO(3), the Special Orthogonal Group. However, singularities can occur where at certain orientations, Euler rotations can no longer uniquely represent all possible rotations. This is a mathematical representation of gimbal lock. This issue would be avoided if, say on the Apollo spacecraft, a third gimbal was added, providing a fourth degree of freedom. [8, §1-2 p32-33] In the 19th century, the mathematician William Rowan Hamilton was interested in the complex numbers for their property that multiplying them causes a plane rotation. For years, he tried to extend the complex numbers to an analogous three-dimensional system, using the basis $1, i, j$ where $i^2 = j^2 = -1$, but could not find a consistent value for ij. [5, §2.B, p8-9] On the 16th of October 1843, Hamilton was walking along Broome Bridge, Ireland, when he had an epiphany: instead of a three-dimensional system, why not try a four-dimensional one? Using the basis $1, i, j, k$, he carved the associated basis laws into the bridge, and a plaque remains today. [5, §2.B, p8]

Both the Apollo spacecraft gimbals and Hamilton's revelation highlight that a fourth dimension is crucial to defining rotations using a non-singular parametrisation, thus avoiding gimbal lock. The main intent of this paper is to show that quaternions of the form QPQ^{-1} where P is a pure quaternion and Q is a unit quaternion represent three-dimensional rotations, and provide a geometric explanation for why this parametrisation avoids gimbal lock.

2 The Algebraic Structure of $\mathbb{H}$

2.1 Definition and Operations

In order to prove that QPQ^{-1} represents a three-dimensional rotation for a pure quaternion P and unit quaternion Q, we must have that every non-zero quaternion Q has a multiplicative inverse. In order to do this, we show that $\mathbb{H}$ - the set of all quaternions, named after Hamilton - is a division ring.

Definition 2.1 (Division Ring)**.** A division ring is a ring in which there exists a multiplicative inverse for each non-zero element. [11, §2.1, p10]

The basic algebra of quaternions is already heavily documented in the literature; instead, in this section we build a toolkit in order to prove this vital and rare property of $\mathbb{H}$. A fitting place to start is, naturally, the definition of a quaternion!

Definition 2.2 (Quaternion)**.** A quaternion, $Q \in \mathbb{H}$, is defined as $Q = w + ix + jy + kz$ such that $w, x, y, z \in \mathbb{R}$ and the symbols i, j, k satisfy the quaternion basis laws. [7, Book 2, Chapter 1, §10, p160]

Definition 2.3 (Quaternion Basis Laws)**.** For a quaternion $w+ix+jy+kz \in \mathbb{H}$, the basis $1, i, j, k$ satisfies the following:

1. $i^2 = j^2 = k^2 = ijk = -1$. As such, the three multiples of these basis terms are called the imaginary parts of the quaternion.

2. $ij = k$, $jk = i$, $ki = j$.

3. $ji = -k$, $kj = -i$, $ik = -j$.

[7, Book 2, Chapter 1, §10, p157-158]

R·C	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

Table 1: Multiplication Table for i,j,k.

Remark. Despite Hamilton's original definition, it is now commonplace notation to write $Q = s + xi + yj + zk$ - notice that the scalar coefficient and basis element of each imaginary term have swapped order. [13, §6.2, p77], [14, §1.2, p2]

In our desired conjugation QPQ^{-1}, P is a pure quaternion. We take this opportunity to define these too.

Definition 2.4 (Pure Quaternion)**.** A pure quaternion is a quaternion with zero scalar part, i.e. $P = 0 + xi + yj + zk$ for $P \in \mathbb{H}$. [2, §2, p137]

With quaternions defined, our next tools of interest are quaternion addition and multiplication. These operations and their well-definedness are essential to proving the division ring structure of $\mathbb{H}$.

Definition 2.5 (Quaternion Operations)**.** For quaternions
$P = s_1 + x_1 i + y_1 j + z_1 k$ and $Q = s_2 + x_2 i + y_2 j + z_2 k$ we have the well-defined operations:

- $P + Q = (s_1 + s_2) + (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$.
- $\lambda P = \lambda s_1 + \lambda x_1 i + \lambda y_1 j + \lambda z_1 k$ for some $\lambda \in \mathbb{R}$.
- $PQ = (s_1 + x_1 i + y_1 j + z_1 k)(s_2 + x_2 i + y_2 j + z_2 k) =$
  $(s_1 s_2 - x_1 x_2 - y_1 y_2 - z_1 z_2) + (s_1 x_2 + x_1 s_2 + y_1 z_2 - z_1 y_2)i$
  $+(s_1 y_2 - x_1 z_2 + y_1 s_2 + z_1 x_2)j + (s_1 z_2 + x_1 y_2 - y_1 x_2 + z_1 s_2)k$.

[4, §1, p4]

Quaternion addition and scalar multiplication are component-wise operations and as such are very simple to work with. However, in its current form, quaternion multiplication is quite unwieldy. Thankfully, we can mitigate this complexity with a useful new trick.

2.1.1 Alternate Forms of Quaternions

Quaternions can take a variety of forms, such as the sum of a scalar and multiple of a unit vector or even a matrix as we shall utilise later on. However, in computationally heavy applications such as computer graphics and animation, the ordered pair form is commonplace (as seen in [10], [13]).

The ordered pair form embeds the quaternion basis laws into Cartesian vector space, meaning that we can use the real orthogonal basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ instead of the imaginary i, j, k basis. Furthermore, it becomes valid to use the Cartesian vector dot and cross products; this drastically reduces the complexity of quaternion multiplication. We adopt this form due to its simplicity and relevance to the application of quaternions to rotations.

Proposition 2.6 (Ordered Pair Form). *Let $Q = s + xi + yj + zk \in \mathbb{H}$. Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x, y, z) \in \mathbb{R}^3$. Then Q can be written as the ordered pair $(s, \mathbf{v})$.*

Corollary 2.7 (Quaternion Multiplication, Revisited). *Let $P = (s_1, \mathbf{v_1})$ and $Q = (s_2, \mathbf{v_2})$ with $P, Q \in \mathbb{H}$. Then:*

1. *The product $PQ = (s_1, \mathbf{v_1})(s_2, \mathbf{v_2}) = (s_1 s_2 - \mathbf{v_1} \cdot \mathbf{v_2},\ s_1\mathbf{v_2} + s_2\mathbf{v_1} + \mathbf{v_1} \times \mathbf{v_2})$.*
2. *The basis $(1, \mathbf{0}), (0, \mathbf{i}), (0, \mathbf{j}), (0, \mathbf{k})$ satisfies the quaternion basis laws.*

Proof. The prior is a long but simple derivation from the original multiplication definition. [10, §5.2, p79] [13, §6.3, p83]. Rodman gives an equivalent result in matrix form. [11, §2.1, p10]
The latter is a proof utilising said bases and the newly defined multiplication operation. We prove each law here.

- $i^2 = -1$: Consider $i = [0, \mathbf{i}]$. Then $i^2 = [0, \mathbf{i}][0, \mathbf{i}] = [0 - \mathbf{i} \cdot \mathbf{i},\ 0 + 0 + \mathbf{i} \times \mathbf{i}] = [0 - 1, 0] = [-1, 0] = -1$ as needed. Similarly for $[0, \mathbf{j}]$ and $[0, \mathbf{k}]$.
- $ij = k$: Consider $i = [0, \mathbf{i}], j = [0, \mathbf{j}]$. Then $ij = [0, \mathbf{i}][0, \mathbf{j}] = [0 - \mathbf{i} \cdot \mathbf{j},\ 0 + 0 + \mathbf{i} \times \mathbf{j}] = [0 - 0, \mathbf{k}] = [0, \mathbf{k}] = k$ as needed.
  Similarly for $[0, \mathbf{j}][0, \mathbf{k}]$,$[0, \mathbf{k}][0, \mathbf{i}]$ and the three negations.
- $ijk = -1$: Consider $i = [0, \mathbf{i}], j = [0, \mathbf{j}], k = [0, \mathbf{k}]$.
  Then $ijk = [0, \mathbf{i}][0, \mathbf{j}][0, \mathbf{k}] = [0, \mathbf{k}][0, \mathbf{k}] = -1$ as needed.

[13, Adapted, §6.3.1, p84] □

2.2 Quaternion Structure

Equipped with a notion of what quaternions are, two vital operations and a way to simplify working, it is time to discern the algebraic structure of $\mathbb{H}$. We at last start building towards our division ring proof!

We begin with a fairly standard result.

Lemma 2.8. *The set $\mathbb{H}$, equipped with quaternion addition and multiplication, is a non-commutative ring.*

Proof. The following properties are explicitly useful in the coming geometry; the remaining properties are stated and proved in the appendix.

1. Closure under Multiplication - Recall Corollary 2.7. The product PQ can also be written in ordered pair form, and as such is a quaternion.

2. Associativity of Multiplication - For all $P, Q, R \in \mathbb{H}$, we have $(PQ)R = P(QR)$. [14, §1.2, p3]

3. Distributivity of Multiplication over Addition - For all $P, Q, R \in \mathbb{H}$, we have $P(Q + R) = PQ + PR$ and $(P + Q)R = PR + QR$. [11, §2.1, p9]

4. Non-Commutative Multiplication - The product $PQ = (s_1 s_2 - \mathbf{v_1} \cdot \mathbf{v_2}, s_1\mathbf{v_2} + s_2\mathbf{v_1} + \mathbf{v_1} \times \mathbf{v_2}) \neq (s_2 s_1 - \mathbf{v_2} \cdot \mathbf{v_1}, s_2\mathbf{v_1} + s_1\mathbf{v_2} + \mathbf{v_2} \times \mathbf{v_1})$ since the cross product is not commutative. [4, Adapted, §1, p5], [1, §1.1, p4].

□

Remark. Bresar's work [1] greatly emphasises the non-commutativity property as it made $\mathbb{H}$ the first non-commutative algebraic system widely utilised, and certainly the first non-commutative division algebra.

Despite being non-commutative, the ring structure of $\mathbb{H}$ allows for considerable algebraic flexibility. In particular, multiplicative associativity ensures that QPQ^{-1} is well-defined provided Q^{-1} is, which remains our goal to show. For now, we set this result aside to discuss a new type of quaternion: the conjugate.

2.3 Division Ring

Definition 2.9 (Conjugate Quaternion)**.** For $Q = s + xi + yj + zk = (s, \mathbf{v})$, its conjugate is defined as $Q^* = s - xi - yj - zk = (s, -\mathbf{v})$. [11, §2.1, p9]

Conjugate quaternions admit the following fascinating result - a special case of commutativity in a non-commutative ring.

Lemma 2.10. *For $Q \in \mathbb{H}$, $QQ^* = Q^*Q$. [14, §1.2, p3]*

Proof. The product of a quaternion and its conjugate $QQ^* = (s^2 - (\mathbf{v} \cdot -\mathbf{v}), s(-\mathbf{v}) + s\mathbf{v} + (\mathbf{v} \times -\mathbf{v})) = (s^2 - (\mathbf{v} \cdot -\mathbf{v}), \mathbf{0}) = (s^2 - (-\mathbf{v} \cdot \mathbf{v}), s\mathbf{v} + s(-\mathbf{v}) + (-\mathbf{v} \times \mathbf{v})) = Q^*Q$. □

Remark. Observe that $QQ^* = Q^*Q = (s^2 - (\mathbf{v} \cdot -\mathbf{v}), \mathbf{0}) = s^2 + x^2 + y^2 + z^2$, a non-negative scalar. Clearly, $QQ^* = Q^*Q = 0$ if and only if $Q = Q^* = 0$. This quantity will be of great importance when developing the geometry of quaternions. For now, however, we return our attention to the above lemma.

This may appear to be an abrupt mathematical turn - we were just discussing algebraic structure - but this seemingly innocuous construction is the key to proving all non-zero quaternions admit a multiplicative inverse. Using this, we find that $\mathbb{H}$ indeed forms a division ring.

Theorem 2.11. *The set of quaternions $\mathbb{H}$ forms a division ring.*

Proof. We have already shown that $\mathbb{H}$ is a ring; we are yet to show the existence of a multiplicative inverse for each non-zero element of $\mathbb{H}$.

We have that $QQ^* = Q^*Q$. Further, $Q^*Q = 0$ if and only if $Q = Q^* = 0$, which is untrue by assumption. Dividing both sides by Q^*Q yields $\frac{QQ^*}{Q^*Q} = 1$ (division of a non-zero scalar by itself). Consequently, $Q^{-1} = \frac{1}{Q^*Q}Q^*$, where $\frac{1}{Q^*Q}$ is a positive scalar. As such, for each non-zero $Q \in \mathbb{H}$, there exists the multiplicative inverse $Q^{-1} = \frac{Q^*}{Q^*Q}$.

Therefore $\mathbb{H}$ is a division ring. [11, Adapted, §2.1, p10] □

Remark. This is a rare result! In fact, Frobenius' Theorem states that any finite dimensional real division algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. This property truly makes quaternions stand out from other types of object. [1, §1.1, p4]

Thanks to the division ring algebraic structure of $\mathbb{H}$, we know that every non-zero quaternion admits a multiplicative inverse. This yields a crucial piece of information: we can carry out conjugation by any non-zero quaternion.

With the division ring structure of $\mathbb{H}$ established, we can begin developing the geometry of quaternions. The algebraic concepts discussed here will no doubt be useful in investigating the relationship between pure quaternions and three-dimensional space.

3 Pure Quaternions and Three-Dimensional Space

Recall that our main goal is to prove how quaternions of the form QPQ^{-1} for Q unit and P pure can describe rotations in three-dimensional space. We have taken a vital first step by showing the existence of a multiplicative inverse for all non-zero quaternions. Our next goal is to show that properties of rotations in three-dimensional space are satisfied by the transformation of pure quaternions QPQ^{-1}.

3.1 The Special Orthogonal Group

Before developing any quaternion geometry, it is prudent to review the properties of a rotation in three-dimensional space in order to show these properties hold for QPQ^{-1}.

Definition 3.1 (Special Orthogonal Group, SO(3))**.** The group SO(3) is composed of orthogonal matrices $M \in \mathbb{R}^{3\times3}$ with determinant 1. These matrices represent length-preserving, orientation-preserving linear transformations. Three-dimensional rotations are defined as elements of this group. [9, §2, p156]

Consequently, in order for QPQ^{-1} with a pure quaternion P and unit quaternion Q to represent a three-dimensional rotation, it must be linear, length-preserving and orientation-preserving. To understand the analogues of these properties in $\mathbb{H}$, we begin to develop essential quaternion geometry, beginning with a notion of length.

3.2 Quaternion Norm and Unit Quaternions

Definition 3.2 (Quaternion Norm)**.** The norm of a quaternion, $\|Q\| = \sqrt{s^2+x^2+y^2+z^2} = \sqrt{QQ^*} = \sqrt{s^2+(\mathbf{v}\cdot\mathbf{v})}$ is a measure of its length, i.e. the distance from (0,0,0,0) to (s,x,y,z). [11, §2.1, p9]

Using the norm, we can at last define the unit quaternion we have seen throughout this essay as part of the conjugation QPQ^{-1}:

Definition 3.3 (Unit Quaternion)**.** A quaternion $Q \in \mathbb{H}$ is a unit quaternion if its norm $\|Q\| = 1$. Similarly to $\mathbb{R}^3$, a unit quaternion Q can be obtained from a quaternion Q' through the equation $Q = \frac{Q'}{\|Q'\|}$. [13, Adapted, §6.14, p92][1]

Remark. Recall that the multiplicative inverse, Q^{-1} of a quaternion Q is $\frac{Q^*}{Q^*Q}$ which we now know equals $\frac{Q^*}{\|Q\|^2}$. Subsequently, a unit quaternion's multiplicative inverse is equal to its conjugate. This idea is linked to why Q must be unit in our target conjugation.

Now we have defined the quaternion norm, recall that our current objective is to prove that QPQ^{-1} is a rotation, i.e., is a member of SO(3). In order to prove the length-preservation property, we show that the quaternion norm is multiplicative.

Proposition 3.4. *The conjugate of a quaternion product* $(PQ)^* = Q^*P^*$. *[4, §1, p6]*

[1] Note that Vince calls this object a normalised quaternion in his definition, unlike most. The name 'unit quaternion' describes a different object in his work.

Proof. Adopting ordered pair form,
$PQ = (s_1s_2 - \mathbf{v_1} \cdot \mathbf{v_2},\ s_1\mathbf{v_2} + s_2\mathbf{v_1} + \mathbf{v_1} \times \mathbf{v_2})$.
Its conjugate $(PQ)^* = (s_1s_2 - \mathbf{v_1} \cdot \mathbf{v_2},\ -s_1\mathbf{v_2} - s_2\mathbf{v_1} - \mathbf{v_1} \times \mathbf{v_2})$
$= (s_2s_1 - ((-\mathbf{v_2}) \cdot (-\mathbf{v_1})), s_2(-\mathbf{v_1}) + s_1(-\mathbf{v_2}) + (-\mathbf{v_2}) \times (-\mathbf{v_1})) = Q^*P^*$. □

Lemma 3.5. *The quaternion norm is multiplicative, i.e* $\|PQ\| = \|P\| \cdot \|Q\|$*.*

Proof. We have $\|PQ\|^2 = PQ(PQ)^* = PQQ^*P^* = P(\|Q\|^2)P^* = PP^*\|Q\|^2 = \|P\|^2\|Q\|^2$. Both sides are non-negative, so square rooting yields the desired result. [11, Adapted, §2.1, p10] □

This lemma is essential to the proof of the length-preservation property of QPQ^{-1} in the next section.
Before directly proving our main result, we are faced with a geometric inconsistency to address: the quaternions, $\mathbb{H}$, form a four-dimensional vector space, yet the rotations we desire are of a three-dimensional vector space, $\mathbb{R}^3$. Therefore, a property of a transformation of $\mathbb{H}$ may not necessarily imply the same property in $\mathbb{R}^3$. In particular, the quaternion norm being preserved by QPQ^{-1} does not imply the three-dimensional Euclidean norm is. This is precisely why QPQ^{-1} must be a transformation of pure quaternions to be a rotation.

Lemma 3.6. *The image (via isomorphism) in* $\mathbb{R}^3$ *of a norm-preserving transformation of pure quaternions preserves the three-dimensional Euclidean norm.*

Proof. Let $\mathbb{P}$ be the subspace of $\mathbb{H}$ consisting of pure quaternions. As a three-dimensional vector space on the real numbers, $\mathbb{P}$ is isomorphic to $\mathbb{R}^3$; we use the natural isomorphism $f : \mathbb{P} \to \mathbb{R}^3$, $xi + yj + zk \mapsto (x, y, z)$. [6, Adapted, §9, p14-15].
For a pure quaternion $P = xi + yj + zk$, we have $\|P\| = \sqrt{x^2 + y^2 + z^2}$, the Euclidean norm of $f(P)$. Consequently, the quaternion norm restricts to the three-dimensional Euclidean norm on $\mathbb{P}$. Therefore, if a transformation of pure quaternions is norm-preserving, so is its image via this isomorphism in $\mathbb{R}^3$. □

Consequently, a linear, norm-preserving, orientation-preserving transformation of pure quaternions indeed corresponds to an element of SO(3). Now having proven select geometric properties of the pure quaternions, we have the tools to show that QPQ^{-1} for Q unit and P pure fulfils these properties and represents a three-dimensional rotation.

4 Quaternion Transformations as Three-Dimensional Rotations

Knowing what properties constitute a rotation in $\mathbb{R}^3$, it is time to prove our main result: that the transformation of pure quaternions QPQ^{-1} for P pure and Q unit represents an element of SO(3), or equivalently, a three-dimensional rotation.

Elements of SO(3) are matrices; being able to represent QPQ^{-1} as a matrix will help prove the relationship between the two objects.

4.1 Matrix Form of QPQ^{-1}

Proposition 4.1 (Matrix form of quaternion conjugation)**.** *Let P, Q be quaternions with P pure. Let the transformation $T_{Q,Q^*}P := QPQ^*$. Then T_{Q,Q^*} is given by the matrix:*

$$\begin{bmatrix} (s_Q)^2+(x_Q)^2+(y_Q)^2+(z_Q)^2 & 0 & 0 & 0 \\ 0 & (s_Q)^2+(x_Q)^2-(y_Q)^2-(z_Q)^2 & -2s_Qz_Q+2x_Qy_Q & 2s_Qy_Q+2x_Qz_Q \\ 0 & 2s_Qz_Q+2x_Qy_Q & (s_Q)^2-(x_Q)^2+(y_Q)^2-(z_Q)^2 & -2s_Qx_Q+2y_Qz_Q \\ 0 & -2s_Qy_Q+2x_Qz_Q & 2s_Qx_Q+2y_Qz_Q & (s_Q)^2-(x_Q)^2-(y_Q)^2+(z_Q)^2 \end{bmatrix}$$

Proof. This matrix can be derived using left and right quaternion multiplication matrices, discussed in Rodman's work. [11, §2.2, p11] We omit the derivation here for brevity. □

As a sanity check, we show that QPQ^{-1} for P pure and Q unit is indeed a transformation of pure quaternions (i.e. the image of $T_{Q,Q^{-1}}P$ is also a pure quaternion).

Proposition 4.2. *The product $T_{Q,Q^{-1}}P$ is a transformation of pure quaternions with P pure, Q unit.*

Proof. Recall that the multiplicative inverse of a unit quaternion is its conjugate. Therefore it is valid to treat the above matrix defining T_{Q,Q^*} as if it were $T_{Q,Q^{-1}}$ in this case. Note that since Q is unit, the first entry of the matrix becomes 1. Carrying out the matrix calculation yields $T_{Q,Q^{-1}}(0, x_P, y_P, z_P)^T = (0, x_{P'}, y_{P'}, z_{P'})^T$ where $P' = x_{P'}i + y_{P'}j + z_{P'}k$ is a pure quaternion. [10, Adapted, §5.3, p82] □

Remark. As the only non-zero element in the first row and column of $T_{Q,Q^{-1}}$ is the leading 1, and the scalar part of P and P' is a fixed 0, we can reduce the transformation to the 3×3 block in the bottom right of $T_{Q,Q^{-1}}$, acting on the vector part of P. We keep the name $T_{Q,Q^{-1}}$ for this reduced matrix.

$$\begin{bmatrix} (s_Q)^2+(x_Q)^2-(y_Q)^2-(z_Q)^2 & -2s_Qz_Q+2x_Qy_Q & 2s_Qy_Q+2x_Qz_Q \\ 2s_Qz_Q+2x_Qy_Q & (s_Q)^2-(x_Q)^2+(y_Q)^2-(z_Q)^2 & -2s_Qx_Q+2y_Qz_Q \\ -2s_Qy_Q+2x_Qz_Q & 2s_Qx_Q+2y_Qz_Q & (s_Q)^2-(x_Q)^2-(y_Q)^2+(z_Q)^2 \end{bmatrix}$$

Now that we have shown that the transformation of pure quaternions QPQ^{-1} can be represented by $T_{Q,Q^{-1}}P$, we can begin to assess its properties.

4.2 QPQ^{-1} is a Three-Dimensional Rotation

4.2.1 QPQ^{-1} is Linear

Recall that in order to be a rotation, this transformation must be linear, length-preserving, and orientation-preserving. We begin by showing linearity.

Lemma 4.3. *The product QPQ^{-1} with P pure and Q unit is a linear transformation of P.*

Proof. Consider $T_{Q,Q^{-1}}(\lambda P_1 + \mu P_2)$ for some scalars $\lambda, \mu \in \mathbb{R}$. Quaternion multiplication is distributive over addition, so $T_{Q,Q^{-1}}(\lambda P_1 + \mu P_2) = Q(\lambda P_1 + \mu P_2)Q^{-1} = (\lambda QP_1 + \mu QP_2)Q^{-1} = \lambda(QP_1Q^{-1}) + \mu(QP_2Q^{-1}) = \lambda T_{Q,Q^{-1}}(P_1) + \mu T_{Q,Q^{-1}}(P_2)$ as needed. □

Therefore, QPQ^{-1} represents a linear transformation of a three-dimensional object, just as an element of SO(3) does.

4.2.2 QPQ^{-1} Preserves Distance

Next, we use the multiplicativity of the quaternion norm to check that QPQ^{-1} is a Euclidean-norm-preserving transformation.

Lemma 4.4. *The length scale factor of QPQ^{-1} is $\|Q\| \cdot \|Q^{-1}\|$. In particular, QPQ^{-1} is a length-preserving transformation of P for P pure and Q unit.*

Proof. Recall that the quaternion norm is multiplicative, i.e. $\|PQ\| = \|P\| \cdot \|Q\|$. Therefore $T_{Q,Q^{-1}}P = QPQ^{-1}$ has quaternion norm $\|T_{Q,Q^{-1}}P\| = \|QPQ^{-1}\| = \|Q\| \cdot \|P\| \cdot \|Q^{-1}\|$. Consequently the quaternion norm of P, $\|P\|$, is scaled by the scalar $\|Q\| \cdot \|Q^{-1}\|$.
We are interested in the case where Q is unit and P is pure. Then the image $T_{Q,Q^{-1}}P$ is also pure, and as such its quaternion norm restricts to the three-dimensional Euclidean norm. When Q is unit, so is Q^{-1}, and thus the Euclidean norm is scaled by 1. Therefore the transformation QPQ^{-1} for Q unit and P pure is a Euclidean-norm-preserving transformation. [13, Adapted, §8.3.3, p140] □

Remark. We have now shown that QPQ^{-1} for P pure and Q unit represents a linear, length-preserving transformation. Together, these two properties imply that the matrix $T_{Q,Q^{-1}}$ is orthogonal, a property of elements of SO(3). While beyond the scope of this essay, Triola provides a tidy proof of this. [12, §1.2, p3]

4.2.3 QPQ^{-1} Preserves Orientation

We have established that QPQ^{-1} for P pure and Q unit is linear, and preserves the three-dimensional Euclidean norm. However, this is not yet enough to classify this transformation as a rotation; a reflection satisfies these criteria too. The key difference is that while a rotation is orientation-preserving, a reflection is orientation-reversing. The matrices representing these transformations have determinant $+1$ and -1 respectively (the only possible determinants of an orthogonal matrix [12, §1.2, p3]).

In order for QPQ^{-1} to be orientation-preserving, its matrix representation $T_{Q,Q^{-1}}$ must have determinant $+1$. While this can be calculated through brute force, we can apply concepts of topology to the quaternions and the determinant function to understand its behaviour for this matrix.

The main concept we will use is the link between continuous functions and connected sets.

Lemma 4.5. *Let E be a connected set. If there exists a continuous function $f : E \to \mathbb{R}^n$, then the range of f is connected.*

Proof. Let $x, y \in E$. Since E is connected, there is a continuous function $g : [0, 1] \to E$ such that $g(0) = x, g(1) = y$. Then the composition $f \circ g : [0, 1] \to \mathbb{R}^n$ is continuous at $(f \circ g)(0) = f(x), (f \circ g)(1) = f(y)$. So $f(E)$ is path-connected, and therefore connected. [11, §3.10, p60] □

We see that this lemma applies quite simply to non-zero quaternions (a connected set) and the determinant of $T_{Q,Q^{-1}}$ (a continuous function of the components of Q). Using this, we prove that the determinant of $T_{Q,Q^{-1}}$ is $+1$ for all non-zero quaternions Q, and thus, $T_{Q,Q^{-1}}$ is orientation-preserving for all non-zero Q.

Theorem 4.6. *The determinant of $T_{Q,Q^{-1}}$ is $+1$ for all non-zero quaternions Q. In particular, transformations of the form QPQ^{-1} for Q unit and P pure are orientation-preserving.*

Proof. Since the set of non-zero quaternions $\mathbb{H} \setminus \{0\}$ is connected, and the function $\det T_{Q,Q^{-1}} : (s_Q, x_Q, y_Q, z_Q) \to \mathbb{R}$ for $(s_Q, x_Q, y_Q, z_Q) \in \mathbb{H} \setminus \{0\}$ is continuous, the possible values of $\det T_{Q,Q^{-1}}$ form a connected set. Therefore, $\det T_{Q,Q^{-1}}$ equals either $+1$ for all non-zero Q or -1 for all non-zero Q.

Substituting $Q = 1$ yields $T_{Q,Q^{-1}} = I$ which has determinant $+1$. Consequently $\det T_{Q,Q^{-1}} = +1$ for all $\mathbb{H} \setminus \{0\}$, so all transformations of the form QPQ^{-1} for Q unit and P pure are orientation-preserving. [11, Adapted, §2.2, p12] □

With this theorem, we have proven that the transformation QPQ^{-1} for P pure and Q unit demonstrates every property of a three-dimensional rotation. We formally show this through the lens of its matrix representation $T_{Q,Q^{-1}}$ and the group SO(3).

Theorem 4.7. *The matrix $T_{Q,Q^{-1}}$ representing the transformation QPQ^{-1}, is an element of SO(3) for a unit quaternion Q and a pure quaternion P.*

Proof.

1. 3×3 Matrix: The restriction of P to pure and Q to unit enables the reduction of the 4×4 matrix T_{Q,Q^*} to the 3×3 matrix $T_{Q,Q^{-1}}$. (Remark of Proposition 4.2)

2. Linear: $T_{Q,Q^{-1}}$ represents a linear transformation, and thus fixes the origin. (Lemma 4.3)

3. Orthogonal: $T_{Q,Q^{-1}}$ represents a Euclidean-norm-preserving transformation, and thus is orthogonal. (Lemma 4.4 and its remark)

4. Determinant $+1$: $T_{Q,Q^{-1}}$ represents an orientation-preserving transformation, and thus has determinant $+1$ (Theorem 4.6)

As a result, $T_{Q,Q^{-1}}$ representing QPQ^{-1} for P pure and Q unit is an element of SO(3), and therefore defines a three-dimensional rotation. □

Consequently, any quaternion product of the form QPQ^{-1} where P is pure and Q is unit defines a three-dimensional rotation. We will conclude by discussing the benefits of representing a rotation in this manner.

5 Conclusion

By proving that QPQ^{-1} where P is pure and Q is unit represents a distance-preserving, orientation-preserving linear transformation of pure quaternions, and that this has an orthogonal matrix representation with determinant $+1$, we have shown that there is a correspondence between products of this form and elements of SO(3). As a result, such transformations represent three-dimensional rotations.

Being able to represent a three-dimensional rotation in the form QPQ^{-1} using quaternions is a significant result with a clear application: avoiding gimbal lock.

Geometrically, the aforementioned Euler rotations system means that any three-dimensional rotation is composed of a sequence of three rotations about the individual axes. Utilising a sequence of such transformations yields the opportunity for two axes of rotation to align during intermediate steps, and the system subsequently loses a degree of freedom. This phenomenon is gimbal lock.

However, using quaternions, we can define a three-dimensional rotation as a single object using the form above. Crucially, this is not a sequence of rotations; in particular, it does not rely on the non-coplanarity of its rotation axes. Therefore, there is no opportunity for gimbal lock to occur. [3, §5, p10]

The non-singular nature of this parametrisation of SO(3) can be further investigated using quaternion calculus to show that there are no points where the Jacobian matrix for this parametrisation loses rank. [3, §5, p10] Furthermore, practical use of this form of rotations can be explored by deriving Rodrigues' rotation formula using this product of quaternions. [13, §7.7, 8.3.3, p119, 139].

For now however, we can rest easy knowing that the films we watch can be produced a little more simply, and that brave astronauts can explore our Universe a little more safely, thanks to quaternions.

A Appendix

Lemma A.1. *The set $\mathbb{H}$, equipped with quaternion addition and multiplication, is a non-commutative ring.*

Proof. The following properties are yet to be proven.

1. Closure under Addition - for $P, Q \in \mathbb{H}$ as in Proposition 2.2, $P + Q = (s_1 + s_2) + (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k \in \mathbb{H}$ as needed.
2. Associativity of Addition - $(P+Q)+R = P+(Q+R)$ for all $P, Q, R \in \mathbb{H}$.
3. Additive Identity - consider $0 = 0 + 0i + 0j + 0k \in \mathbb{H}$. Then for all $P \in \mathbb{H}$, $P + 0 = 0 + P = P$.
4. Additive Inverses - for any $A = s + xi + yj + zk \in \mathbb{H}$, consider $-A = -s - xi - yj - zk \in \mathbb{H}$. Then $A + (-A) = (-A) + A = s - s + xi - xi + yj - yj + zk - zk = 0$.
5. Commutativity of Addition - $P + Q = (s_1 + s_2) + (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k = (s_2 + s_1) + (x_2 + x_1)i + (y_2 + y_1)j + (z_2 + z_1)k = Q + P$.
6. Multiplicative Identity - consider $1 = 1 + 0i + 0j + 0k \in \mathbb{H}$. Then for all $P \in \mathbb{H}$, $P1 = 1P = P$.

   Properties 1 to 5 from [10, §5.2, p80].

$\square$

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